Principles of Mathematics 12

Geometric Series

$$\sum_{k=4}^{7}(3k+2)^2$$

LESSON 2

Sum Formulas, Sigma Notation, and Applications

Principles of Math 12

EXPLAINED!

By
Barry
Mabillard

SUM FORMULAS: GIVEN A GEOMETRIC SERIES, THE FOLLOWING FORMULA CAN BE USED TO FIND THE SUM OF ANY NUMBER OF TERMS.

$$S_N = \frac{A(1-R^N)}{1-R}$$

S_N = THE SUM OF N TERMS
A = FIRST TERM
R = COMMON RATIO
N = NUMBER OF TERMS TO BE SUMMED.

Example 1: Given the series 6 + 12 + 24 +..., determine the sum of the first 13 terms.

a = 6
r = 2
n = 13

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{13} = \frac{6(1-2^{13})}{1-2}$$

$$S_{13} = 49146$$

Example 2: Given the series $\frac{5}{4}, \frac{5}{6}, \frac{5}{9}, \ldots$, how many terms are required to have a sum of 3.7066

Step 1: First find the common ratio.

$$r = \frac{5}{6} \div \frac{5}{4}$$

$$r = \frac{5}{6} \times \frac{4}{5}$$

$$r = \frac{4}{6}$$

$$r = \frac{2}{3}$$

Step 2: Now state what you know.

$$a = \frac{5}{4} = 1.25$$

$$r = \frac{2}{3} = 0.667$$

$$S_n = 3.7066$$

Step 3: Finally, do the calculation.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$3.7066 = \frac{1.25(1-0.667^n)}{1-0.667}$$

$$3.7066 = \frac{1.25(1-0.667^n)}{0.333} \rightarrow \text{Cross Multiply}$$

$$1.2355 = 1.25(1-0.667^n)$$

$$0.9884 = 1-0.667^n \rightarrow \text{Subtract 1 from both sides}$$

$$-0.01156 = -0.667^n \rightarrow \text{Cancel out the negatives}$$

$$0.01156 = 0.667^n$$

$$\log 0.01156 = \log(0.667)^n \rightarrow \text{Log both sides to isolate n}$$

$$\log 0.01156 = n \log 0.667$$

$$n = \frac{\log 0.01156}{\log 0.667}$$

$$n = 11$$

GIVEN A GEOMETRIC SERIES, THE FOLLOWING FORMULA CAN BE USED TO FIND THE SUM WHEN THE LAST TERM IN THE SERIES IS KNOWN.

$$S_N = \frac{RT_N - A}{R - 1}$$

S_N = THE SUM OF N TERMS
A = FIRST TERM
R = COMMON RATIO
T_N = VALUE OF THE LAST TERM

<u>Example 3:</u> Given the series 6 + 12 + 24 +... 384, determine the sum of the series.

$$S_n = \frac{rt_n - a}{r - 1}$$

$$S_n = \frac{(2)(384) - 6}{2 - 1}$$

$$S_n = 762$$

THE FOLLOWING FORMULA CAN BE USED TO FIND A PARTICULAR TERM IF SUMS ARE KNOWN.

$$T_N = S_N - S_{N-1}$$

T_N = TERM VALUE
S_N = SUM OF N TERMS
S_{N-1} = SUM OF N-1 TERMS

<u>Example 4:</u> If the sum of the first 7 terms in a geometric series is 45, and the sum of the first 8 terms is 56, determine the value of the eighth term.

$$t_n = S_n - S_{n-1}$$

$$t_8 = S_8 - S_7$$

$$t_8 = 56 - 45$$

$$t_8 = 11$$

**<u>Example 5:</u> The sum of 3 terms is 34, and the fourth term is 15.
Determine the sum of 4 terms.**

$$t_n = S_n - S_{n-1}$$

$$t_4 = S_4 - S_3$$

$$15 = S_4 - 34$$

$$S_4 = 49$$

QUESTIONS:

1) Find the sum of the following series:
a) $2 + 4 + 8 + 16 + \ldots$ S_9

b) $3 + 3\sqrt{3} + 9 \ldots$ S_7

c) $12 + 3 + \frac{3}{4} + \ldots$ S_{11}

d) $\frac{5}{3} + \frac{10}{9} + \frac{20}{27} + \ldots$ S_7

2) Find the sum of the following series:
a) $4 + 8 + 16 + \ldots 1024$

b) $81 + 27 + 9 + \ldots \frac{1}{9}$

c) $\frac{2}{5} + \frac{4}{5} + \frac{8}{5} + \ldots \frac{32768}{5}$

3) a) If the sum of the first 7 terms in a geometric series is 89, and the sum of the first 8 terms is 104, determine the value of the eighth term.

b) The sum of 3 terms is 56, and the fourth term is 25. Determine the sum of 4 terms.

c) The sum of a series is given by the formula $S(n) = 3^n(n-2)$. Determine the value of the 5^{th} term.

ANSWERS:

1) a) 1022
b) 187.55
c) 16
d) 4.707

2) a) 2044
b) 121.44
c) 13106.8

3) a) $t_8 = S_8 - S_7$
$t_8 = 104 - 89$
$t_8 = 15$
b) $t_4 = S_4 - S_3$
$25 = S_4 - 56$
$S_4 = 81$
c) $t_5 = S_5 - S_4$
$t_5 = 729 - 162$
$t_5 = 567$

SIGMA NOTATION: THIS IS USED TO SHOW A SERIES IN CONDENSED FORM. EXPAND BY PLUGGING IN THE BOTTOM NUMBER, THEN KEEP PLUGGING IN CONSECUTIVE NUMBERS UNTIL THE TOP NUMBER IS REACHED.

Example 1: **Expand** $\sum_{n=2}^{5} 3n - 4$

Expand by substituting 2, 3, 4, & 5 for n, then add the results.

$= [3(2)-4]+[3(3)-4]+[3(4)-4]+[3(5)-4]$

$= 2+5+8+11$

$= 26$

GIVEN: $\sum_{k=3}^{8} k+2$

THE TOTAL NUMBER OF TERMS IS FOUND USING THE FORMULA:
(*Top Number – Bottom Number*) + 1

In the example above, the number of terms is
(8 – 3) + 1 = 6

Example 2: **Expand** $\sum_{n=2}^{4} 3\left(\frac{1}{2}\right)^{n-1}$

Expand by substituting 2, 3, & 4 for n, then add the results.

$= \left[3\left(\frac{1}{2}\right)^{2-1}\right]+\left[3\left(\frac{1}{2}\right)^{3-1}\right]+\left[3\left(\frac{1}{2}\right)^{4-1}\right]$

$= \left[3\left(\frac{1}{2}\right)\right]+\left[3\left(\frac{1}{2}\right)^{2}\right]+\left[3\left(\frac{1}{2}\right)^{3}\right]$

$= \frac{3}{2}+\frac{3}{4}+\frac{3}{8}$

$= \frac{21}{8}$

Example 3: **Determine the sum of:** $\sum_{k=5}^{13} 4(2)^{k-2}$

If the series is expanded, the a & r values can be easily read: $\sum_{k=5}^{13} 4(2)^{k-2} = 32+64+128+\ldots$

a = 32

r = 2

Total number of terms = (13 – 5) + 1 = 9

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{32(1-2^9)}{1-2}$$

$$S_n = 16532$$

Example 4: Write the series $5+1+\frac{1}{5}\ldots\frac{1}{78125}$ in sigma notation

Step 1: First determine the last term in the series.

$t_n = ar^{n-1}$

$\frac{1}{78125} = 5\left(\frac{1}{5}\right)^{n-1}$

$\frac{1}{78125} = 5\left(\frac{1}{5}\right)^{n-1}$ → Divide both sides by 5

$\frac{1}{390625} = \left(\frac{1}{5}\right)^{n-1}$

$\left(\frac{1}{5}\right)^{8} = \left(\frac{1}{5}\right)^{n-1}$

$8 = n - 1$

$n = 9$

Step 2: State what you know:

$a = 5$

$r = \frac{1}{5}$

$n = 9$

Step 3: Now express as sigma. The expression inside sigma is simply the general term: ar^{n-1}

Answer $= \sum_{n=1}^{9} 5\left(\frac{1}{5}\right)^{n-1}$

Sometimes diploma questions will ask you to simplify the exponents in sigma notation:

Ex 1:

$$\sum_{n=1}^{5} 2(2)^{n-1} = \sum_{n=1}^{5} 2^{(1+n-1)} = \sum_{n=1}^{5} 2^{n}$$

Ex 2:

$$\sum_{n=1}^{5} \frac{1}{3}(3)^{n-1} = \sum_{n=1}^{5} \frac{3^{n-1}}{3} = \sum_{n=1}^{5} 3^{(n-1-1)} = \sum_{n=1}^{5} 3^{n-2}$$

Ex 3:

$$\sum_{n=1}^{9} 5\left(\frac{1}{5}\right)^{n-1} = \sum_{n=1}^{9} \frac{5}{5^{n-1}} = \sum_{n=1}^{9} 5^{[1-(n-1)]} = \sum_{n=1}^{9} 5^{2-n}$$

Example 5: Write the series $20+40+80+\ldots+163840$ in sigma notation:

Step 1: First determine the last term in the series.

$t_n = ar^{n-1}$

$163840 = 20(2)^{n-1}$

$8192 = (2)^{n-1}$

$2^{13} = 2^{n-1}$

$13 = n - 1$

$n = 14$

Step 2: State what you know:

$a = 20$

$r = 2$

$n = 14$

Step 3: Express as sigma.

Answer $= \sum_{n=1}^{14} 20(2)^{n-1}$

* This cannot be simplified further since there is no common base.

Geometric Series: Lesson 2
Part II – Sigma Notation

QUESTIONS:

1) Expand and evaluate the following:

a) $\sum_{n=2}^{5} 2n-1$

b) $\sum_{n=1}^{3} 3\left(\frac{1}{3}\right)^{n-1}$

c) $\sum_{n=1}^{3} 2^{n^2-1}$

2) Determine the sum using **formulas.** *(Rather than simply adding all terms)*

a) $\sum_{k=3}^{8} 3\left(\frac{1}{2}\right)^{k+1}$

b) $\sum_{n=7}^{17} 2^{n}$

c) $\sum_{k=4}^{12} 8\left(\frac{1}{2}\right)^{k-2}$

3) Simplify the following so only a single base is present:

a) $\sum_{n=1}^{7} 5(5)^{n-2}$

b) $\sum_{n=1}^{8} \frac{1}{4}(4)^{n+3}$

c) $\sum_{n=1}^{12} 3\left(\frac{1}{3}\right)^{n-1}$

4) Write the following series in sigma notation *(single base if possible),* then find the sum.

a) $4+1+\frac{1}{4}...\frac{1}{1024}$

b) $15+45+135+...+295245$

c) $1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{8192}$

ANSWERS:

1) a) 24 **b)** $\frac{13}{3}$ **c)** 265

2) a) $\frac{189}{512}$ **b)** 262016 **c)** $\frac{511}{128}$

3) a) $\sum_{n=1}^{7} 5^{n-2}$ **b)** $\sum_{n=1}^{8} 4^{n+2}$ **c)** $\sum_{n=1}^{12} 3^{2-n}$

4) a) $\sum_{n=1}^{7} 4\left(\frac{1}{4}\right)^{n-1} = \sum_{n=1}^{7} 4^{2-n} = \frac{5461}{1024}$ **b)** $\sum_{n=1}^{10} 15(3)^{n-1} = 442860$ **c)** $\sum_{n=1}^{14} \left(\frac{1}{2}\right)^{n-1} = 2$

INFINITE SERIES: A SERIES THAT GOES ON FOREVER IS CALLED AN INFINITE SERIES.

3 + 6 + 12 + 24 + . . . + 384 → THIS IS A FINITE SERIES, SINCE IT HAS A HIGHEST NUMBER.

3 + 6 + 12 + 24 + . . . → THIS IS AN INFINITE SERIES, SINCE THERE IS NO HIGHEST NUMBER.

When finding the sum of an infinite series, two different outcomes are possible:

Outcome 1: If the common ratio is between -1 and +1, an actual number can be obtained for the sum. This is called a convergent series, and the formula used to find this sum is $S_\infty = \frac{a}{1-r}$

Outcome 2: If the common ratio is less than -1 or bigger than 1, it is impossible to obtain a sum for an infinite series. This is called a divergent series.

Remember The Following:

A common ratio between -1 and +1 can be written mathematically as

$-1 < r < 1$ or $|r| < 1$

A common ratio less than -1 or bigger than 1 can be written mathematically as

$|r| > 1$

Example 1: Find the sum of the series 6 + 12 + 24 +...

The common ratio is 2, so this is a divergent series and no sum can be found.

Example 2: Find the sum of the series $-1 + \frac{1}{2} - \frac{1}{4} + \ldots$

Step 1: First state the known information:

$a = -1$

$r = -\frac{1}{2}$

Step 2: Since the common ratio is between -1 and +1, the sum is convergent and the formula can be used.

$$S_\infty = \frac{-1}{1-\left(-\frac{1}{2}\right)} = \frac{-1}{1+\frac{1}{2}} = \frac{-1}{\frac{3}{2}} = -\frac{2}{3}$$

Example 3: Find the sum of the series $\sum_{k=1}^{\infty} 100(0.3)^{k-1}$ to the nearest hundredth.

Step 1: First state the known information:

$a = 100$

$r = 0.3$

Step 2: Since the common ratio is between -1 and +1, the sum is convergent and the formula can be used.

$$S_\infty = \frac{100}{1-0.3} = \frac{100}{0.7} = 142.86$$

QUESTIONS:

1) Find the sum of the following series:

a) $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$

b) $2 + 2\sqrt{3} + 6 + \ldots$

c) $-12 + 3 - \frac{3}{4} + \ldots$

d) $\frac{5}{3} + \frac{10}{9} + \frac{20}{27} + \ldots$

2) Find the sum of the following series:

a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

b) $\sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{2}{3}\right)^{n-1}$

c) $\sum_{n=1}^{\infty} 2(3)^{n-1}$

3) The sum of an infinite geometric series is $\frac{13}{5}$ and the common ratio is $-\frac{1}{4}$. Determine the first term.

4) An infinite geometric series has the terms $t_1 = 32$ and $t_4 = 4$. Determine the sum of the infinite series.

ANSWERS:

1) **a)** 4 **b)** No Sum (Divergent) **c)** -9.6 **d)** 5

2) **a)** 6 **b)** $\frac{12}{5}$ **c)** No Sum (Divergent)

3) $\frac{13}{4}$

4) 64

Example 1: A chessboard has 64 squares. If one penny is placed on the first square, then doubled to two pennies on the second, then doubled to four pennies on the third, how much money will be on the board when the 64th square is reached?

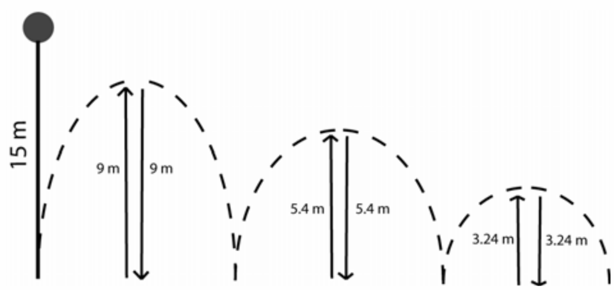
a = 0.01
r = 2
n = 64

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{64} = \frac{0.01((2)^{64} - 1)}{2 - 1}$$

$$S_{64} = \$1.84 \times 10^{17} = \$184,000,000,000,000,000$$

Example 2: A ball is dropped from a height of 15 m and bounces to 60% of the previous height. How far has the ball traveled when it hits the ground for the fourth time?



a = 15
r = 0.6
n = 4

You can start this question by plugging the numbers directly into the sum formula.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_4 = \frac{15(1 - 0.6^4)}{1 - 0.6}$$

$$S_4 = 32.64 \text{ m}$$

The sum just calculated, by plugging in numbers directly from the question, is **NOT THE ANSWER!**

You need to do two extra things to get the answer:

1) Since the ball goes up and down, multiply the sum by 2 to get the total distance.

2) Now subtract 15, since the first height only has a down motion.

The total distance is **50.28 m**

Example 3: A ball is dropped from a height of 15 m and bounces to 60% of the previous height. How far has the ball traveled when it comes to rest?
Use the infinite series formula.

$$S_\infty = \frac{a}{1 - r} = \frac{15}{1 - 0.6} = 37.5$$

Now multiply by 2, then subtract the initial height to obtain the total distance of 60 m.

QUESTIONS:

1) A gameboard has 32 squares. If one nickel is placed on the first square, then doubled to two nickels on the second, then doubled to four nickels on the third, how much money will be on the board when the 32^{nd} square is reached?

2) A ball is dropped from a height of 17 m and bounces to 68% of the previous height. How far has the ball traveled vertically when it hits the ground for the eighth time?

3) A ball is dropped from a height of 20 m and bounces to 4/5 of the previous height.
a) How far has the ball traveled vertically when it hits the ground for the tenth time?

b) How far has the ball traveled vertically when it comes to rest?

ANSWERS:

1) $214748364.40
2) 84.4
3) **a)** 158.53 m
b) 180 m