

Principles of Mathematics 12

TRIGONOMETRY II

$$(\cos x - 1)(\tan x - 1) = 0$$

LESSON TWELVE

Sum & Difference Identities

Principles of
Math 12
EXPLAINED!
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TRIGONOMETRY LESSON 12

PART 1 EXPANDING SUM & DIFFERENCE

Sum & Difference Identities: The following formulas are used to expand trigonometric functions that have addition & subtraction in brackets.

$\sin(A + B) \neq \sin A + \sin B$, so we must use these rules whenever we want to expand.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Example 1: Expand $\sin(60^\circ - 45^\circ)$

$$\begin{aligned}\sin(60^\circ - 45^\circ) &= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Example 2: Expand $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$$\begin{aligned}\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \left(\frac{\sqrt{6}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

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PART I EXPANDING SUM & DIFFERENCE

Find the exact value by expanding each of the following:

1) $\sin(45^\circ + 60^\circ)$

5) $\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

2) $\cos(45^\circ - 30^\circ)$

6) $\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

3) $\sin(60^\circ - 135^\circ)$

7) $\cos\left(0 - \frac{3\pi}{4}\right)$

4) $\cos(150^\circ + 45^\circ)$

8) $\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

TRIGONOMETRY LESSON 12

PART I EXPANDING SUM & DIFFERENCE

$$\begin{aligned}
 1. \quad & \sin(45^\circ + 60^\circ) \\
 &= \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ \\
 &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\
 &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
 &= \frac{\sqrt{2} + \sqrt{6}}{4}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \cos(45^\circ - 30^\circ) \\
 &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
 &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
 &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \sin(60^\circ - 135^\circ) \\
 &= \sin 60^\circ \cos 135^\circ - \cos 60^\circ \sin 135^\circ \\
 &= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
 &= \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
 &= \frac{-\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \cos(150^\circ + 45^\circ) \\
 &= \cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ \\
 &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
 &= \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
 &= \frac{-\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) \\
 &= \sin \frac{\pi}{2} \cos \frac{\pi}{3} - \cos \frac{\pi}{2} \sin \frac{\pi}{3} \\
 &= (1)\left(\frac{1}{2}\right) - (0)\left(\frac{\sqrt{3}}{2}\right) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\
 &= \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\
 &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
 &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \cos\left(0 - \frac{3\pi}{4}\right) \\
 &= \cos 0 \cos \frac{3\pi}{4} + \sin 0 \sin \frac{3\pi}{4} \\
 &= (1)\left(-\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) \\
 &= -\frac{\sqrt{2}}{2}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) \\
 &= \cos \frac{\pi}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{2} \sin \frac{\pi}{3} \\
 &= (0)\left(\frac{1}{2}\right) - (1)\left(\frac{\sqrt{3}}{2}\right) \\
 &= -\frac{\sqrt{3}}{2}
 \end{aligned}$$

TRIGONOMETRY LESSON 12

PART II CONDENSING SUM & DIFFERENCE

Given an expanded form, we can work backwards to find a single trigonometric expression that can be easily solved using the unit circle.

Example 1: Express $\sin 85^\circ \cos 5^\circ + \cos 85^\circ \sin 5^\circ$ as a single trigonometric expression and solve.

We know $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Looking at the question and matching to the sum formula, we can see that:

$$A = 85^\circ \text{ and } B = 5^\circ$$

Now plug A & B into the *left* side of the formula:

$$\begin{aligned}\sin(A + B) &= \sin(85^\circ + 5^\circ) \\ &= \sin(90^\circ) \\ &= 1\end{aligned}$$

Example 2: Express $\frac{1}{\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)}$ as a single trigonometric expression and solve.

We know: $\sin(A - B) = \sin A \cos B - \cos A \sin B$

We have from the denominator of the question: $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$

Comparing the two, $A = \frac{\pi}{3}$ & $B = \frac{\pi}{6}$

Plug A & B into $\sin(A - B)$

$$\begin{aligned}&= \sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\ &= \sin\left(\frac{2\pi}{6} - \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{6}\right)\end{aligned}$$

Subtract radians by finding a common denominator. (Or think in terms of degrees.)

We now know the denominator is $\sin\left(\frac{\pi}{6}\right)$.

Thus, we have $\frac{1}{\sin\left(\frac{\pi}{6}\right)} = \csc\left(\frac{\pi}{6}\right)$. Solving, $\frac{1}{\sin\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{1}{2}} = 2$

TRIGONOMETRY LESSON 12

PART II CONDENSING SUM & DIFFERENCE

For each of the following, express as a single trigonometric expression and solve using the unit circle.

1) $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$

2) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

3)
$$\frac{1}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)}$$

4)
$$\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{3}\right)$$

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PART II CONDENSING SUM & DIFFERENCE

$$5) \frac{1}{\cos(-15^\circ)\cos(30^\circ) + \sin(-15^\circ)\sin(30^\circ)}$$

$$6) \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$$

$$7) \frac{1}{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)}$$

$$8) \cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{2}\right)$$

TRIGONOMETRY LESSON 12

PART II CONDENSING SUM & DIFFERENCE

1. $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$
 $A = 60^\circ$ & $B = 15^\circ$
 $\cos(60^\circ - 15^\circ)$
 $\cos(45^\circ)$
 $= \frac{\sqrt{2}}{2}$

2. $\cos \frac{\pi}{6} \cos \frac{\pi}{6} - \sin \frac{\pi}{6} \sin \frac{\pi}{6}$
 $A = \frac{\pi}{6}$ & $B = \frac{\pi}{6}$
 $\cos\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$
 $= \cos\left(\frac{2\pi}{6}\right)$
 $= \cos\left(\frac{\pi}{3}\right)$
 $= \frac{1}{2}$

3. $\frac{1}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)}$
 $A = \frac{\pi}{2}$ & $B = \frac{\pi}{3}$
 $\frac{1}{\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)}$
 $= \csc\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$
 $= \csc\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right)$ Getting A Common Denominator
 $= \csc\left(\frac{\pi}{6}\right)$
 $= 2$

4. $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{3}\right)$
 $A = \frac{5\pi}{12}$ & $B = \frac{\pi}{3}$
 $\sin\left(\frac{5\pi}{12} + \frac{\pi}{3}\right)$
 $= \sin\left(\frac{5\pi}{12} + \frac{4\pi}{12}\right)$
 $= \sin\left(\frac{9\pi}{12}\right)$
 $= \sin\left(\frac{3\pi}{4}\right)$
 $= \frac{\sqrt{2}}{2}$

5. $\frac{1}{\cos(-15^\circ)\cos(30^\circ) + \sin(-15^\circ)\sin(30^\circ)}$
 $A = -15^\circ$ & $B = 30^\circ$
 $\frac{1}{\cos(-15^\circ - 30^\circ)}$
 $= \frac{1}{\cos(-45^\circ)}$
 $= \frac{1}{\frac{\sqrt{2}}{2}}$
 $= \frac{2}{\sqrt{2}}$
 $= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
 $= \sqrt{2}$

6. $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$
 $A = \frac{\pi}{3}$ & $B = \frac{\pi}{6}$
 $\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$
 $= \sin\left(\frac{2\pi}{6} - \frac{\pi}{6}\right)$
 $= \sin\frac{\pi}{6}$
 $= \frac{1}{2}$

7. $\frac{1}{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)}$
 $A = \frac{\pi}{2}$ & $B = \frac{\pi}{4}$
 $\frac{1}{\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)}$
 $= \sec\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$
 $= \sec\left(\frac{2\pi}{4} - \frac{\pi}{4}\right)$
 $= \sec\left(\frac{\pi}{4}\right)$
 $= \frac{2}{\sqrt{2}}$
 $= \sqrt{2}$

8. $\cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{2}\right)$
 $A = \frac{2\pi}{3}$ & $B = \frac{\pi}{2}$
 $\cos\left(\frac{2\pi}{3} - \frac{\pi}{2}\right)$
 $= \cos\left(\frac{4\pi}{6} - \frac{3\pi}{6}\right)$
 $= \cos\frac{\pi}{6}$
 $= \frac{\sqrt{3}}{2}$

TRIGONOMETRY LESSON 12

PART III NON UNIT CIRCLE ANGLES

The sum & difference formulas are useful in determining the exact values of sine & cosine for angles not on the unit circle.

Example 1: Find the exact value of $\sin 15^\circ$

First, think of how you can get 15° by using angles on the unit circle:

$$15^\circ = 60^\circ - 45^\circ$$

$$15^\circ = 45^\circ - 30^\circ$$

$$15^\circ = 135^\circ - 120^\circ$$

$$15^\circ = -30^\circ + 45^\circ$$

As you can see, there are many possibilities, you can choose any of them and still get the right answer.

We'll use the top one, $15^\circ = 60^\circ - 45^\circ$

$$\begin{aligned} \sin 15^\circ &= \sin(60^\circ - 45^\circ) \\ &= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

Example 2: Find the exact value of $\sec\left(-\frac{5\pi}{12}\right)$

It's easiest to think in terms of degrees, so convert: $-\frac{5\pi}{12} \times \frac{180^\circ}{\pi} = -75^\circ$ ($-75^\circ = -45^\circ - 30^\circ$)

$$\begin{aligned} \sec(-75^\circ) &= \sec(-45^\circ - 30^\circ) \\ &= \frac{1}{\cos(-45^\circ - 30^\circ)} \\ &= \frac{1}{\cos(-45^\circ)\cos(30^\circ) + \sin(-45^\circ)\sin(30^\circ)} \\ &= \frac{1}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)} \\ &= \frac{1}{\frac{\sqrt{6} - \sqrt{2}}{4}} \\ &= \frac{4}{\sqrt{6} - \sqrt{2}} \\ &= \frac{4}{\sqrt{6} - \sqrt{2}} \end{aligned}$$

Now rationalize the denominator of your answer.

$$\begin{aligned} &\frac{4}{\sqrt{6} - \sqrt{2}} \\ &= \frac{4}{\sqrt{6} - \sqrt{2}} \times \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} + \sqrt{2}} \\ &= \frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} \\ &= \frac{4(\sqrt{6} + \sqrt{2})}{4} \\ &= \sqrt{6} + \sqrt{2} \end{aligned}$$

TRIGONOMETRY LESSON 12

PART III NON UNIT CIRCLE ANGLES

Find the exact value of each of the following:

Note that there are multiple ways of getting to the correct answer. Rationalize the denominator when necessary.

1) $\cos(-15^\circ)$

2) $\sec(105^\circ)$

3) $\csc(-105^\circ)$

4) $\sin\left(-\frac{5\pi}{12}\right)$

5) $\csc(165^\circ)$

6) $\sec\left(\frac{13\pi}{12}\right)$

TRIGONOMETRY LESSON 12

PART III NON UNIT CIRCLE ANGLES

$$\begin{aligned}
 1. \quad & \cos(30^\circ - 45^\circ) \\
 &= \cos 30^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ \\
 &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
 &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \sec(60^\circ + 45^\circ) \\
 &= \frac{1}{\cos(60^\circ + 45^\circ)} \\
 &= \frac{1}{\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ} \\
 &= \frac{1}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} \\
 &= \frac{1}{\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}} \\
 &= \frac{1}{\frac{\sqrt{2} - \sqrt{6}}{4}} \\
 &= \frac{4}{\sqrt{2} - \sqrt{6}} \\
 &= \frac{4(\sqrt{2} - \sqrt{6})}{2 - 6} \\
 &= \frac{4(\sqrt{2} - \sqrt{6})}{-4} \\
 &= -1(\sqrt{2} - \sqrt{6}) \\
 &= \sqrt{6} - \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \csc(-60^\circ - 45^\circ) \\
 &= \frac{1}{\sin(-60^\circ - 45^\circ)} \\
 &= \frac{1}{\sin(-60^\circ) \cos 45^\circ - \cos(-60^\circ) \sin 45^\circ} \\
 &= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} \\
 &= \frac{1}{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}} \\
 &= \frac{1}{-\frac{\sqrt{6} + \sqrt{2}}{4}} \\
 &= \frac{4}{-\sqrt{6} - \sqrt{2}} \\
 &= \frac{4(-\sqrt{6} + \sqrt{2})}{6 - 2} \\
 &= \frac{4(-\sqrt{6} + \sqrt{2})}{4} \\
 &= -\sqrt{6} + \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \sin\left(-\frac{5\pi}{12}\right) \\
 &= \sin(-75^\circ) \\
 &= \sin(-45^\circ - 30^\circ) \\
 &= \sin(-45^\circ) \cos 30^\circ - \cos(-45^\circ) \sin 30^\circ \\
 &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
 &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
 &= \frac{-\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \csc(165^\circ) \\
 &= \frac{1}{\sin(165^\circ)} \\
 &= \frac{1}{\sin(120^\circ + 45^\circ)} \\
 &= \frac{1}{\sin(120^\circ) \cos 45^\circ + \cos(120^\circ) \sin 45^\circ} \\
 &= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} \\
 &= \frac{1}{\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}} \\
 &= \frac{1}{\frac{\sqrt{6} - \sqrt{2}}{4}} \\
 &= \frac{4}{\sqrt{6} - \sqrt{2}} \\
 &= \frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} \\
 &= \frac{4(\sqrt{6} + \sqrt{2})}{4} \\
 &= \sqrt{6} + \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \sec\left(\frac{13\pi}{12}\right) \\
 &= \sec(195^\circ) \\
 &= \sec(150^\circ + 45^\circ) \\
 &= \frac{1}{\cos(150^\circ + 45^\circ)} \\
 &= \frac{1}{\cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ} \\
 &= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} \\
 &= \frac{1}{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}} \\
 &= \frac{1}{-\frac{\sqrt{6} + \sqrt{2}}{4}} \\
 &= \frac{4}{-(\sqrt{6} + \sqrt{2})} \\
 &= -\frac{4}{\sqrt{6} + \sqrt{2}} \times \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} \\
 &= \frac{-4(\sqrt{6} - \sqrt{2})}{6 - 2} \\
 &= \frac{-4(\sqrt{6} - \sqrt{2})}{4} \\
 &= -(\sqrt{6} - \sqrt{2})
 \end{aligned}$$

TRIGONOMETRY LESSON 12

PART IV SUM & DIFFERENCE PROOFS

The following examples illustrate some basic proofs you can do with the sum & difference identities:

Example 1: Prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

We start with the formula: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Plug in $A = \frac{\pi}{2}$ & $B = x$

$$\cos\left(\frac{\pi}{2} - x\right) = \cos A \cos B + \sin A \sin B$$

$$= \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x$$

$$= (0) \cos x + (1) \sin x$$

$$= \sin x$$

Example 2: Prove that $\csc(\pi + x) = -\csc x$

$$\csc(\pi + x)$$

$$= \frac{1}{\sin(\pi + x)}$$

$$= \frac{1}{\sin \pi \cos x + \cos \pi \sin x}$$

$$= \frac{1}{(0) \cos x + (-1) \sin x}$$

$$= \frac{1}{-\sin x}$$

$$= -\csc x$$

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PART IV SUM & DIFFERENCE PROOFS

Prove each of the following:

1) $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$

5) $\sin\left(\frac{\pi}{2} - x\right) =$

2) $\sin(270^\circ - x) = -\cos x$

6) $\csc\left(\frac{\pi}{2} + x\right) =$

3) $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

7) $\sec(\pi + x)$

4) $\cos(\pi - x) =$

8) $\csc(\pi - x)$

TRIGONOMETRY LESSON 12

PART IV SUM & DIFFERENCE PROOFS

1. $\cos\left(\frac{3\pi}{2} - x\right)$

$$= \cos \frac{3\pi}{2} \cos x + \sin \frac{3\pi}{2} \sin x$$

$$= (0) \cos x + (-1) \sin x$$

$$= -\sin x$$

2. $\sin(270^\circ - x)$

$$= \sin 270^\circ \cos x - \cos 270^\circ \sin x$$

$$= (-1) \cos x - (0) \sin x$$

$$= -\cos x$$

3. $\cos\left(\frac{\pi}{2} + x\right)$

$$= \cos \frac{\pi}{2} \cos x - \sin \frac{\pi}{2} \sin x$$

$$= (0) \cos x - (1) \sin x$$

$$= -\sin x$$

4. $\cos(\pi - x)$

$$= \cos \pi \cos x + \sin \pi \sin x$$

$$= (-1) \cos x + (0) \sin x$$

$$= -\cos x$$

5. $\sin\left(\frac{\pi}{2} - x\right)$

$$= \sin \frac{\pi}{2} \cos x - \cos \frac{\pi}{2} \sin x$$

$$= (1) \cos x - (0) \sin x$$

$$= \cos x$$

6. $\csc\left(\frac{\pi}{2} + x\right)$

$$= \frac{1}{\sin\left(\frac{\pi}{2} + x\right)}$$

$$= \frac{1}{\sin \frac{\pi}{2} \cos x + \cos \frac{\pi}{2} \sin x}$$

$$= \frac{1}{(1) \cos x + (0) \sin x}$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

7. $\sec(\pi + x)$

$$= \frac{1}{\cos(\pi + x)}$$

$$= \frac{1}{\cos \pi \cos x - \sin \pi \sin x}$$

$$= \frac{1}{(-1) \cos x - (0) \sin x}$$

$$= \frac{1}{-\cos x}$$

$$= -\sec x$$

8. $\csc(\pi - x)$

$$= \frac{1}{\sin(\pi - x)}$$

$$= \frac{1}{\sin \pi \cos x - \cos \pi \sin x}$$

$$= \frac{1}{(0) \cos x - (-1) \sin x}$$

$$= \frac{1}{\sin x}$$

$$= \csc x$$

TRIGONOMETRY LESSON 12

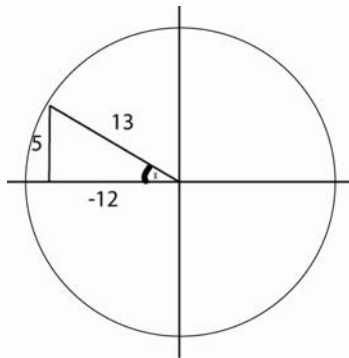
PART V TRIANGLE QUESTIONS

In some questions, you will be given incomplete information, and must use triangles to find all the trig ratios required by the sum & difference formulas.

Example 1: Given $\tan x = -\frac{5}{12}$ (In quadrant II) and

$\tan y = \frac{3}{5}$ (In quadrant III), find the exact value of $\sec(x+y)$

Draw a triangle corresponding to $\tan x = -\frac{5}{12}$ and find the unknown side.

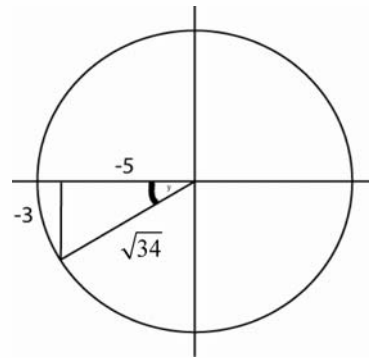


$$\begin{aligned} a^2 + b^2 &= c^2 \\ (-12)^2 + (5)^2 &= c^2 \\ 169 &= c^2 \\ c &= 13 \end{aligned}$$

Now find $\sin x$ & $\cos x$.

$$\begin{aligned} \sin x &= \frac{5}{13} \\ \cos x &= \frac{-12}{13} \end{aligned}$$

Draw a triangle corresponding to $\tan y = \frac{3}{5}$ and find the unknown side.



$$\begin{aligned} a^2 + b^2 &= c^2 \\ (-5)^2 + (-3)^2 &= c^2 \\ 34 &= c^2 \\ c &= \sqrt{34} \end{aligned}$$

Now find $\sin y$ & $\cos y$

$$\begin{aligned} \sin y &= \frac{-3}{\sqrt{34}} \\ \cos y &= \frac{-5}{\sqrt{34}} \end{aligned}$$

$$\begin{aligned} \sec(x+y) &= \frac{1}{\cos(x+y)} \\ &= \frac{1}{\cos x \cos y - \sin x \sin y} \\ &= \frac{1}{\left(\frac{-12}{13}\right)\left(\frac{-5}{\sqrt{34}}\right) - \left(\frac{5}{13}\right)\left(\frac{-3}{\sqrt{34}}\right)} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\frac{60}{13\sqrt{34}} + \frac{15}{13\sqrt{34}}} \\ &= \frac{1}{\frac{75}{13\sqrt{34}}} \\ &= \frac{13\sqrt{34}}{75} \quad \text{Final Answer} \end{aligned}$$

TRIGONOMETRY LESSON 12

PART V TRIANGLE QUESTIONS

$$\sin x = -\frac{1}{2} \quad (\text{Quadrant III})$$

1)

$$\tan y = \frac{3}{4} \quad (\text{Quadrant I})$$

Find $\cos(x - y)$

$$\cos x = \frac{12}{13} \quad (\text{Quadrant IV})$$

2)

$$\csc y = -\frac{5}{2} \quad (\text{Quadrant III})$$

Find $\csc(x + y)$

$$\sec x = \frac{7}{5} \quad (\text{Quadrant I})$$

3)

$$\cot y = -\frac{3}{4} \quad (\text{Quadrant II})$$

Find $\sin(x - y)$

TRIGONOMETRY LESSON 12

PART V TRIANGLE QUESTIONS

$$\tan x = -\frac{6}{7} \quad (\text{Quadrant IV})$$

4)

Find $\sec(x + y)$

$$\sin y = -\frac{2}{5} \quad (\text{Quadrant IV})$$

5)

$$\cot x = -5 \quad (\cos x < 0)$$

Find $\csc(x - y)$

$$\tan y = \frac{3}{4} \quad (\sin y > 0)$$

6)

$$\sec x = -\frac{8}{7} \quad (\sin x > 0)$$

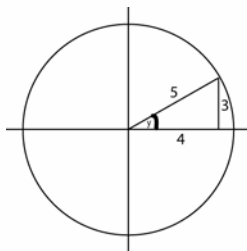
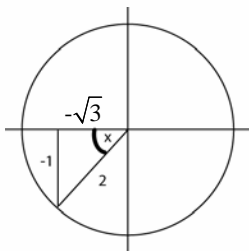
Find $\sec(x - y)$

$$\tan y = -3 \quad (\sin y < 0)$$

TRIGONOMETRY LESSON 12

PART V TRIANGLE QUESTIONS

1) First draw out each triangle, then use Pythagoras to find the unknown side:



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{-1}{2}$$

$$\sin y = \frac{3}{5}$$

$$\cos x = \frac{-\sqrt{3}}{2}$$

$$\cos y = \frac{4}{5}$$

Use these to evaluate $\cos(x - y)$

$$\cos(x - y)$$

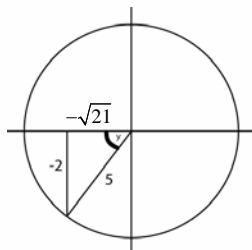
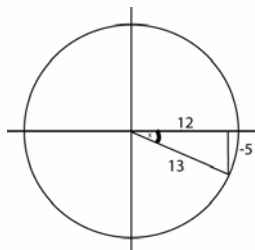
$$= \cos x \cos y + \sin x \sin y$$

$$= \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{4}{5}\right) + \left(\frac{-1}{2}\right)\left(\frac{3}{5}\right)$$

$$= \frac{-4\sqrt{3}}{10} - \frac{3}{10}$$

$$= \frac{-4\sqrt{3} - 3}{10}$$

2) First draw out each triangle, then use Pythagoras to find the unknown side:



Now state the sine cosine trig ratios for each triangle:

$$\sin x = \frac{-5}{13}$$

$$\sin y = \frac{-2}{5}$$

$$\cos x = \frac{12}{13}$$

$$\cos y = \frac{-\sqrt{21}}{5}$$

Use these to evaluate $\csc(x + y)$

$$\csc(x + y)$$

$$= \frac{1}{\sin x \cos y + \cos x \sin y}$$

$$= \frac{1}{\left(\frac{-5\sqrt{21}}{65}\right) + \left(\frac{-24}{65}\right)}$$

$$= \frac{1}{\left(\frac{-5}{13}\right)\left(\frac{-\sqrt{21}}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{-2}{5}\right)}$$

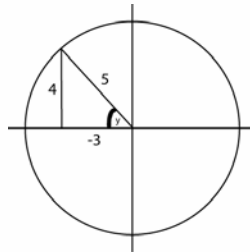
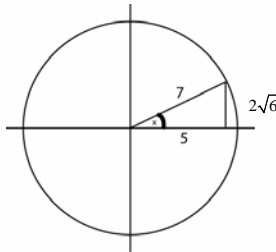
$$= \frac{1}{\frac{-5\sqrt{21} - 24}{65}}$$

$$= \frac{65}{-5\sqrt{21} - 24}$$

TRIGONOMETRY LESSON 12

PART V TRIANGLE QUESTIONS

3) First draw out each triangle, and use Pythagoras to find the unknown side:



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{2\sqrt{6}}{7}$$

$$\cos x = \frac{5}{7}$$

$$\sin y = \frac{4}{5}$$

$$\cos y = \frac{-3}{5}$$

Use these to evaluate $\sin(x - y)$

$$\sin(x - y)$$

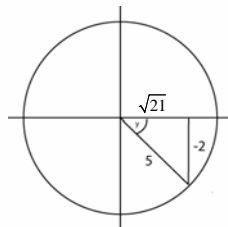
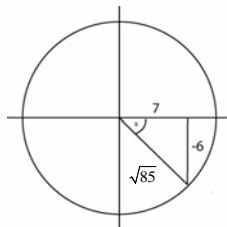
$$= \sin x \cos y - \cos x \sin y$$

$$= \left(\frac{2\sqrt{6}}{7}\right)\left(\frac{-3}{5}\right) - \left(\frac{5}{7}\right)\left(\frac{4}{5}\right)$$

$$= \frac{-6\sqrt{6}}{35} - \frac{20}{35}$$

$$= \frac{-6\sqrt{6} - 20}{35}$$

4) First draw out each triangle, and use Pythagoras to find the unknown side:



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{-6}{\sqrt{85}}$$

$$\cos x = \frac{7}{\sqrt{85}}$$

$$\sin y = \frac{-2}{5}$$

$$\cos y = \frac{\sqrt{21}}{5}$$

Use these to evaluate $\sec(x + y)$

$$\sec(x + y)$$

$$= \frac{1}{\cos x \cos y - \sin x \sin y}$$

$$= \frac{1}{\left(\frac{7}{\sqrt{85}}\right)\left(\frac{\sqrt{21}}{5}\right) - \left(\frac{-6}{\sqrt{85}}\right)\left(\frac{-2}{5}\right)}$$

$$= \frac{1}{\left(\frac{7\sqrt{21}}{5\sqrt{85}}\right) - \left(\frac{12}{5\sqrt{85}}\right)}$$

$$= \frac{1}{\frac{7\sqrt{21} - 12}{5\sqrt{85}}}$$

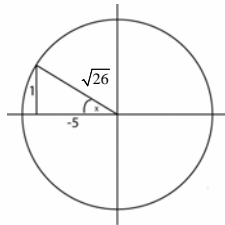
$$= \frac{5\sqrt{85}}{7\sqrt{21} - 12}$$

TRIGONOMETRY LESSON 12

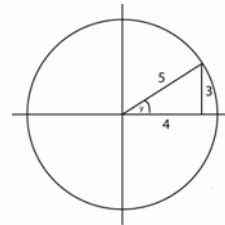
PART V TRIANGLE QUESTIONS

5) First draw out each triangle, and use Pythagoras to find the unknown side:

$\cot x < 0$ in quadrants II & IV
 $\cos x < 0$ in quadrants II & III
 Overlap in II



$\tan y > 0$ in quadrants I & III
 $\sin y > 0$ in quadrants I & II
 Overlap in I



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{1}{\sqrt{26}}$$

$$\cos x = \frac{-5}{\sqrt{26}}$$

$$\sin y = \frac{3}{5}$$

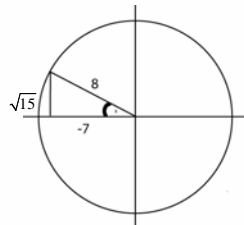
$$\cos y = \frac{4}{5}$$

Use these to evaluate $\csc(x - y)$

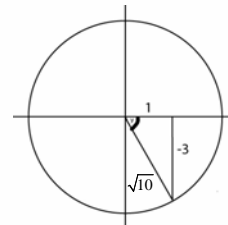
$$\begin{aligned} \csc(x - y) &= \frac{1}{\left(\frac{4}{5\sqrt{26}}\right) - \left(\frac{-15}{5\sqrt{26}}\right)} \\ &= \frac{1}{\sin x \cos y - \cos x \sin y} \\ &= \frac{1}{\left(\frac{1}{\sqrt{26}}\right)\left(\frac{4}{5}\right) - \left(\frac{-5}{\sqrt{26}}\right)\left(\frac{3}{5}\right)} \\ &= \frac{1}{\frac{4}{5\sqrt{26}} + \frac{15}{5\sqrt{26}}} \\ &= \frac{1}{\frac{19}{5\sqrt{26}}} \\ &= \frac{5\sqrt{26}}{19} \end{aligned}$$

6) First draw out each triangle, and use Pythagoras to find the unknown side:

$\sec x < 0$ in quadrants II & III
 $\sin x > 0$ in quadrants I & II
 Overlap in II



$\tan y < 0$ in quadrants II & IV
 $\sin y < 0$ in quadrants III & IV
 Overlap in IV



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{\sqrt{15}}{8}$$

$$\cos x = \frac{-7}{8}$$

$$\sin y = \frac{-3}{\sqrt{10}}$$

$$\cos y = \frac{1}{\sqrt{10}}$$

Use these to evaluate $\sec(x - y)$

$$\begin{aligned} \sec(x - y) &= \frac{1}{\cos x \cos y + \sin x \sin y} \\ &= \frac{1}{\left(\frac{-7}{8}\right)\left(\frac{1}{\sqrt{10}}\right) + \left(\frac{\sqrt{15}}{8}\right)\left(\frac{-3}{\sqrt{10}}\right)} \\ &= \frac{1}{\left(\frac{-7}{8\sqrt{10}}\right) + \left(\frac{-3\sqrt{15}}{8\sqrt{10}}\right)} \\ &= \frac{1}{\frac{-7 - 3\sqrt{15}}{8\sqrt{10}}} \\ &= \frac{8\sqrt{10}}{-7 - 3\sqrt{15}} \end{aligned}$$

TRIGONOMETRY LESSON 12

PART VI DOUBLE ANGLE IDENTITIES

The following double angle identities are frequently used:

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\sin 2A = 2\sin A \cos A \quad \cos 2A = 2\cos^2 A - 1$$

$$\cos 2A = 1 - 2\sin^2 A$$

Example 1: Expand $\sin 60^\circ$ using $\sin 2A = 2\sin A \cos A$:

$$\sin 2A = 2\sin A \cos A$$

$$\sin 60^\circ = 2\sin 30^\circ \cos 30^\circ$$

What goes here is
always half of the
angle on the left side.

Example 2: Expand $\cos 90^\circ$ using $\cos 2A = 2\cos^2 A - 1$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 90^\circ = 2\cos^2 45^\circ - 1$$

Example 3: Expand $\cos \frac{2\pi}{3}$ using $\cos 2A = \cos^2 A - \sin^2 A$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos \frac{2\pi}{3} = \cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$$

Example 4: Expand $\cos 8x$ using $\cos 2A = 1 - 2\sin^2 A$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\cos 8x = 1 - 2\sin^2 4x$$

TRIGONOMETRY LESSON 12

PART VI DOUBLE ANGLE IDENTITIES

In the following examples, you must work backwards to condense the identity:

Example 5: Condense: $1 - 2\sin^2 150^\circ$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\cos 300^\circ = 1 - 2\sin^2 150^\circ$$



What goes on the left side is double the angle on the right.

Example 6: Condense: $2\sin 3x \cos 3x$

$$\sin 2A = 2\sin A \cos A$$

$$\sin 6x = 2\sin 3x \cos 3x$$

Example 7: Condense: $\cos^2 \pi - \sin^2 \pi$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2\pi = \cos^2 \pi - \sin^2 \pi$$

Questions:

1) Expand $\sin 180^\circ$ using $\sin 2A = 2\sin A \cos A$

2) Expand $\cos \frac{\pi}{3}$ using $\cos 2A = 1 - 2\sin^2 A$

TRIGONOMETRY LESSON 12

PART VI DOUBLE ANGLE IDENTITIES

3) Expand $\cos 16x$ using $\cos 2A = \cos^2 A - \sin^2 A$

4) Expand $\cos \frac{\pi}{2}$ using $\cos 2A = 2\cos^2 A - 1$

5) Condense: $2\cos^2 \frac{2\pi}{3} - 1$

6) Condense: $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

7) Condense: $\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x$

TRIGONOMETRY LESSON 12

PART VI DOUBLE ANGLE IDENTITIES

1) $\sin 180^\circ$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin 180^\circ = 2 \sin 90^\circ \cos 90^\circ$$

2) $\cos \frac{\pi}{3}$

$$\cos 2A = 1 - 2 \sin^2 A$$

$$\cos \frac{\pi}{3} = 1 - 2 \sin^2 \frac{\pi}{6}$$

3) $\cos 16x$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 16x = \cos^2 8x - \sin^2 8x$$

4) $\cos \frac{\pi}{2}$

$$\cos 2A = 2 \cos^2 A - 1$$

$$\cos \frac{\pi}{2} = 2 \cos^2 \frac{\pi}{4} - 1$$

5) $2 \cos^2 \frac{2\pi}{3} - 1$

$$\cos 2A = 2 \cos^2 A - 1$$

$$\cos \frac{4\pi}{3} = 2 \cos^2 \frac{2\pi}{3} - 1$$

6) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos \frac{\pi}{3} = \cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$$

7) $\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos x = \cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x$$

TRIGONOMETRY LESSON 12

PART VII DOUBLE ANGLE PROOFS

In the following examples, the double-angle identities will be used in completing proofs.

Example 1: Prove $\sin 2x + \cos x = \cos x(2\sin x + 1)$

$$\begin{aligned}\sin 2x + \cos x \\ &= 2\sin x \cos x + \cos x \\ &= \cos x(2\sin x + 1)\end{aligned}$$

Example 2: Prove: $\cos 2x = \cos^2 x - \sin^2 x$ Hint: Write $\cos 2x$ as $\cos(x + x)$

$$\begin{aligned}\cos(x + x) &= \cos x \cos x - \sin x \sin x \\ \cos(x + x) &= \cos^2 x - \sin^2 x\end{aligned}$$

Questions:

1) $\cos 2x + \cos x = (2\cos x - 1)(\cos x + 1)$ **2)** $2\cos 2x - \sin x + 1 = -(4\sin x - 3)(\sin x + 1)$

3) $\cos 2x = 2\cos^2 x - 1$

4) $\cos 2x = 1 - 2\sin^2 x$

TRIGONOMETRY LESSON 12

PART VII DOUBLE ANGLE PROOFS

5) $\frac{1 + \cos 2x}{\sin 2x} = \cot x$

6) $\frac{2}{1 + \cos 2x} = \sec^2 x$

7) $\sin 2x = 2 \sin x \cos x$

8) $\cos 2x - 1 + 2 \sin x = 2 \sin x(1 - \sin x)$

9) $(\sin x + \cos x)^2 = 1 + \sin 2x$

10) $\sin(x - y)\sin(x + y) = \cos^2 y - \cos^2 x$

TRIGONOMETRY LESSON 12

PART VII DOUBLE ANGLE PROOFS

$$\begin{aligned}
 1) \quad & \cos 2x + \cos x \\
 &= 2\cos^2 x - 1 + \cos x \\
 &= 2\cos^2 x + \cos x - 1 \\
 &= (2\cos x - 1)(\cos x + 1)
 \end{aligned}$$

$$\begin{aligned}
 2) \quad & 2\cos 2x - \sin x + 1 \\
 &= 2(1 - 2\sin^2 x) - \sin x + 1 \\
 &= 2 - 4\sin^2 x - \sin x + 1 \\
 &= -4\sin^2 x - \sin x + 3 \\
 &= -(4\sin^2 x + \sin x - 3) \\
 &= -(4\sin x - 3)(\sin x + 1)
 \end{aligned}$$

$$\begin{aligned}
 3) \quad & \cos 2x = \cos(x + x) \\
 & \cos(x + x) = \cos x \cos x - \sin x \sin x \\
 & \cos 2x = \cos^2 x - \sin^2 x \\
 & \cos 2x = \cos^2 x - (1 - \cos^2 x) \\
 & \cos 2x = \cos^2 x - 1 + \cos^2 x \\
 & \cos 2x = 2\cos^2 x - 1
 \end{aligned}$$

$$\begin{aligned}
 4) \quad & \cos 2x = 1 - 2\sin^2 x \\
 & \cos(x + x) = \cos x \cos x - \sin x \sin x \\
 & \cos 2x = \cos^2 x - \sin^2 x \\
 & \cos 2x = (1 - \sin^2 x) - \sin^2 x \\
 & \cos 2x = 1 - \sin^2 x - \sin^2 x \\
 & \cos 2x = 1 - 2\sin^2 x
 \end{aligned}$$

$$\begin{aligned}
 5) \quad & \frac{1 + \cos 2x}{\sin 2x} \\
 &= \frac{1 + (2\cos^2 x - 1)}{2\sin x \cos x} \\
 &= \frac{2\cos^2 x}{2\sin x \cos x} \\
 &= \frac{\cos x}{\sin x} \\
 &= \cot x
 \end{aligned}$$

$$\begin{aligned}
 6) \quad & \frac{2}{1 + \cos 2x} = \sec^2 x \\
 &= \frac{2}{1 + (2\cos^2 x - 1)} \\
 &= \frac{2}{2\cos^2 x} \\
 &= \frac{1}{\cos^2 x} \\
 &= \sec^2 x
 \end{aligned}$$

$$\begin{aligned}
 7) \quad & \sin 2x = \sin(x + x) \\
 & \sin(x + x) = \sin x \cos x + \cos x \sin x \\
 &= 2\sin x \cos x
 \end{aligned}$$

$$\begin{aligned}
 8) \quad & \cos 2x - 1 + 2\sin x \\
 &= (1 - 2\sin^2 x) - 1 + 2\sin x \\
 &= -2\sin^2 x + 2\sin x \\
 &= 2\sin x(-\sin x + 1) \\
 &= 2\sin x(1 - \sin x)
 \end{aligned}$$

$$\begin{aligned}
 9) \quad & (\sin x + \cos x)^2 \\
 &= \sin^2 x + 2\sin x \cos x + \cos^2 x \\
 &= \sin^2 x + \cos^2 x + 2\sin x \cos x \\
 &= 1 + 2\sin x \cos x
 \end{aligned}$$

$$\begin{aligned}
 10) \quad & \sin(x - y)\sin(x + y) = \cos^2 y - \cos^2 x \\
 &= [\sin x \cos y - \cos x \sin y][\sin x \cos y + \cos x \sin y] \\
 &= (\sin x \cos y)^2 + \sin x \cos y \cos x \sin y - \cos x \sin y \sin x \cos y - (\cos x \sin y)^2 \\
 &= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y \\
 &= \sin^2 x \cos^2 y - (1 - \sin^2 x)(1 - \cos^2 y) \\
 &= \sin^2 x \cos^2 y - [1 - \cos^2 y - \sin^2 x + \sin^2 x \cos^2 y] \\
 &= \sin^2 x \cos^2 y - 1 + \cos^2 y + \sin^2 x - \sin^2 x \cos^2 y \\
 &= \cos^2 y + \sin^2 x - 1 \\
 &= \cos^2 y - (1 - \sin^2 x) \\
 &= \cos^2 y - \cos^2 x
 \end{aligned}$$