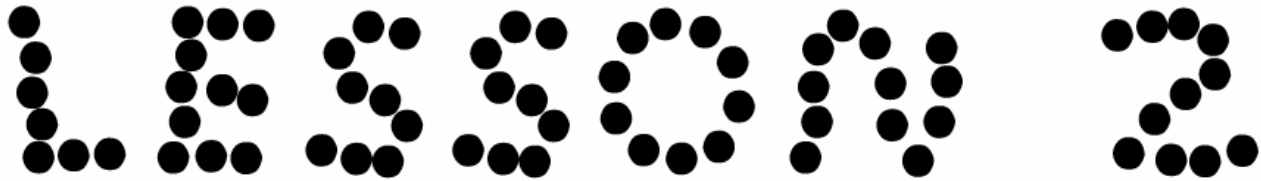


Pure Math 30:

Geometric Series

$$\sum_{k=4}^7 (3k + 2)^2$$



Sum Formulas, Sigma Notation, and Applications

Pure Math
30:

EXPLAINED!

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Geometric Series: Lesson 2

Part I – Sum Formulas

SUM FORMULAS: GIVEN A GEOMETRIC SERIES, THE FOLLOWING FORMULA CAN BE USED TO FIND THE SUM OF ANY NUMBER OF TERMS.

$$S_N = \frac{A(R^N - 1)}{R - 1}$$

S_N = THE SUM OF N TERMS

A = FIRST TERM

R = COMMON RATIO

N = NUMBER OF TERMS TO BE SUMMED.

Example 1: Given the series $6 + 12 + 24 + \dots$, determine the sum of the first 13 terms.

$$a = 6$$

$$r = 2$$

$$n = 13$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{13} = \frac{6(2^{13} - 1)}{2 - 1}$$

$$S_{13} = 49146$$

Example 2: Given the series $\frac{5}{4}, \frac{5}{6}, \frac{5}{9}, \dots$, how many terms are required to have a sum of 3.7066

Step 1: First find the common ratio.

$$r = \frac{5}{6} \div \frac{5}{4}$$

$$r = \frac{5}{6} \times \frac{4}{5}$$

$$r = \frac{4}{6}$$

$$r = \frac{2}{3}$$

Step 2: Now state what you know.

$$a = \frac{5}{4} = 1.25$$

$$r = \frac{2}{3} = 0.667$$

$$S_n = 3.7066$$

Step 3: Finally, do the calculation.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$3.7066 = \frac{1.25((0.667)^n - 1)}{0.667 - 1}$$

$$3.7066 = \frac{1.25((0.667)^n - 1)}{-0.333} \rightarrow \text{Cross Multiply}$$

$$-1.2355 = 1.25((0.667)^n - 1)$$

$$-0.9884 = (0.667)^n - 1 \rightarrow \text{Add 1 to both sides}$$

$$0.01156 = (0.667)^n$$

$$\log 0.01156 = \log (0.667)^n \rightarrow \text{Log both sides to isolate } n$$

$$\log 0.01156 = n \log 0.667$$

$$n = \frac{\log 0.01156}{\log 0.667}$$

$$n = 11$$

PURE MATH 30: EXPLAINED!

Geometric Series: Lesson 2

Part I – Sum Formulas

GIVEN A GEOMETRIC SERIES, THE FOLLOWING FORMULA CAN BE USED TO FIND THE SUM WHEN THE LAST TERM IN THE SERIES IS KNOWN.

$$S_N = \frac{RT_N - A}{R - 1}$$

S_N = THE SUM OF N TERMS
 A = FIRST TERM
 R = COMMON RATIO
 T_N = VALUE OF THE LAST TERM

Example 3: Given the series $6 + 12 + 24 + \dots + 384$, determine the sum of the series.

$$S_n = \frac{rt_n - a}{r - 1}$$

$$S_n = \frac{(2)(384) - 6}{2 - 1}$$

$$S_n = 762$$

THE FOLLOWING FORMULA CAN BE USED TO FIND A PARTICULAR TERM IF SUMS ARE KNOWN.

$$T_N = S_N - S_{N-1}$$

T_N = TERM VALUE
 S_N = SUM OF N TERMS
 S_{N-1} = SUM OF N-1 TERMS

Example 4: If the sum of the first 7 terms in a geometric series is 45, and the sum of the first 8 terms is 56, determine the value of the eighth term.

$$t_n = S_n - S_{n-1}$$

$$t_8 = S_8 - S_7$$

$$t_8 = 56 - 45$$

$$t_8 = 11$$

Example 5: The sum of 3 terms is 34, and the fourth term is 15.

Determine the sum of 4 terms.

$$t_n = S_n - S_{n-1}$$

$$t_4 = S_4 - S_3$$

$$15 = S_4 - 34$$

$$S_4 = 49$$

Geometric Series: Lesson 2

Part I – Sum Formulas

QUESTIONS:

1) Find the sum of the following series:

a) $2 + 4 + 8 + 16 + \dots S_9$

b) $3 + 3\sqrt{3} + 9 \dots S_7$

c) $12 + 3 + \frac{3}{4} + \dots S_{11}$

d) $\frac{5}{3} + \frac{10}{9} + \frac{20}{27} + \dots S_7$

2) Find the sum of the following series:

a) $4 + 8 + 16 + \dots 1024$

b) $81 + 27 + 9 + \dots \frac{1}{9}$

c) $\frac{2}{5} + \frac{4}{5} + \frac{8}{5} + \dots \frac{32768}{5}$

3) a) If the sum of the first 7 terms in a geometric series is 89, and the sum of the first 8 terms is 104, determine the value of the eighth term.

b) The sum of 3 terms is 56, and the fourth term is 25. Determine the sum of 4 terms.

c) The sum of a series is given by the formula $S(n) = 3^n(n-2)$. Determine the value of the 5th term.

ANSWERS:

- 1) a) 1022
b) 187.55
c) 16
d) 4.707

- 2) a) 2044
b) 121.44
c) 13106.8

- 3) a) $t_8 = S_8 - S_7$
 $t_8 = 104 - 89$
 $t_8 = 15$
b) $t_4 = S_4 - S_3$
 $25 = S_4 - 56$
 $S_4 = 81$
c) $t_5 = S_5 - S_4$
 $t_5 = 729 - 162$
 $t_5 = 567$

Geometric Series: Lesson 2

Part II – Sigma Notation

SIGMA NOTATION: THIS IS USED TO SHOW A SERIES IN CONDENSED FORM. EXPAND BY PLUGGING IN THE BOTTOM NUMBER, THEN KEEP PLUGGING IN CONSECUTIVE NUMBERS UNTIL THE TOP NUMBER IS REACHED.

Example 1: Expand $\sum_{n=2}^5 3n - 4$

Expand by substituting 2, 3, 4, & 5 for n, then add the results.

$$\begin{aligned} &= [3(2) - 4] + [3(3) - 4] + [3(4) - 4] + [3(5) - 4] \\ &= 2 + 5 + 8 + 11 \\ &= 26 \end{aligned}$$

Example 2: Expand $\sum_{n=2}^4 3\left(\frac{1}{2}\right)^{n-1}$

Expand by substituting 2, 3, & 4 for n, then add the results.

$$\begin{aligned} &= \left[3\left(\frac{1}{2}\right)^{2-1}\right] + \left[3\left(\frac{1}{2}\right)^{3-1}\right] + \left[3\left(\frac{1}{2}\right)^{4-1}\right] \\ &= \left[3\left(\frac{1}{2}\right)\right] + \left[3\left(\frac{1}{2}\right)^2\right] + \left[3\left(\frac{1}{2}\right)^3\right] \\ &= \frac{3}{2} + \frac{3}{4} + \frac{3}{8} \\ &= \frac{21}{8} \end{aligned}$$

Example 3: Determine the sum of: $\sum_{k=5}^{13} 4(2)^{k-2}$

If the series is expanded, the a & r values can be easily read: $\sum_{k=5}^{13} 4(2)^{k-2} = 32 + 64 + 128 + \dots$

$$a = 32$$

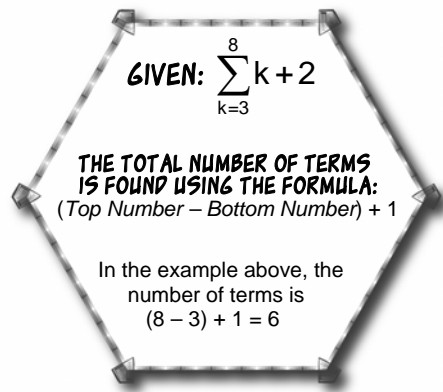
$$r = 2$$

$$\text{Total number of terms} = (13 - 5) + 1 = 9$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_9 = \frac{32((2)^9 - 1)}{2 - 1}$$

$$S_9 = 16352$$



Geometric Series: Lesson 2

Part II – Sigma Notation

Example 4: Write the series $5 + 1 + \frac{1}{5} \dots \frac{1}{78125}$ in sigma notation

Step 1: First determine the last term in the series.

$$t_n = ar^{n-1}$$

$$\frac{1}{78125} = 5 \left(\frac{1}{5} \right)^{n-1}$$

$$\frac{1}{78125} = 5 \left(\frac{1}{5} \right)^{n-1} \rightarrow \text{Divide both sides by 5}$$

$$\frac{1}{390625} = \left(\frac{1}{5} \right)^{n-1}$$

$$\left(\frac{1}{5} \right)^8 = \left(\frac{1}{5} \right)^{n-1}$$

$$8 = n - 1$$

$$n = 9$$

Step 2: State what you know:

$$a = 5$$

$$r = \frac{1}{5}$$

$$n = 9$$

Step 3: Now express as sigma. The expression inside sigma is simply the general term: ar^{n-1}

$$\text{Answer} = \sum_{n=1}^9 5 \left(\frac{1}{5} \right)^{n-1}$$

Sometimes diploma questions will ask you to simplify the exponents in sigma notation:

Ex 1:

$$\sum_{n=1}^5 2(2)^{n-1} = \sum_{n=1}^5 2^{(1+n-1)} = \sum_{n=1}^5 2^n$$

Ex 2:

$$\sum_{n=1}^5 \frac{1}{3}(3)^{n-1} = \sum_{n=1}^5 \frac{3^{n-1}}{3} = \sum_{n=1}^5 3^{(n-1-1)} = \sum_{n=1}^5 3^{n-2}$$

Ex 3:

$$\sum_{n=1}^9 5 \left(\frac{1}{5} \right)^{n-1} = \sum_{n=1}^9 \frac{5}{5^{n-1}} = \sum_{n=1}^9 5^{[1-(n-1)]} = \sum_{n=1}^9 5^{2-n}$$

Example 5: Write the series $20 + 40 + 80 + \dots + 163840$ in sigma notation:

Step 1: First determine the last term in the series.

$$t_n = ar^{n-1}$$

$$163840 = 20(2)^{n-1}$$

$$8192 = (2)^{n-1}$$

$$2^{13} = 2^{n-1}$$

$$13 = n - 1$$

$$n = 14$$

Step 2: State what you know:

$$a = 20$$

$$r = 2$$

$$n = 14$$

Step 3: Express as sigma.

$$\text{Answer} = \sum_{n=1}^{14} 20(2)^{n-1}$$

* This cannot be simplified further since there is no common base.

Geometric Series: Lesson 2

Part II – Sigma Notation

QUESTIONS:

1) Expand and evaluate the following:

a) $\sum_{n=2}^5 2n-1$

b) $\sum_{n=1}^3 3\left(\frac{1}{3}\right)^{n-1}$

c) $\sum_{n=1}^3 2^{n^2-1}$

3) Simplify the following so only a single base is present:

a) $\sum_{n=1}^7 5(5)^{n-2}$

b) $\sum_{n=1}^8 \frac{1}{4}(4)^{n+3}$

c) $\sum_{n=1}^{12} 3\left(\frac{1}{3}\right)^{n-1}$

2) Determine the sum using **formulas**.
(Rather than simply adding all terms)

a) $\sum_{k=3}^8 3\left(\frac{1}{2}\right)^{k+1}$

b) $\sum_{n=7}^{17} 2^n$

c) $\sum_{k=4}^{12} 8\left(\frac{1}{2}\right)^{k-2}$

4) Write the following series in sigma notation (*single base if possible*), then find the sum.

a) $4+1+\frac{1}{4}+\dots+\frac{1}{1024}$

b) $15+45+135+\dots+295245$

c) $1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{8192}$

ANSWERS:

1) a) 24 b) $\frac{13}{3}$ c) 265

2) a) $\frac{189}{512}$ b) 262016 c) $\frac{511}{128}$

3) a) $\sum_{n=1}^7 5^{n-2}$ b) $\sum_{n=1}^8 4^{n+2}$ c) $\sum_{n=1}^{12} 3^{2-n}$

4) a) $\sum_{n=1}^7 4\left(\frac{1}{4}\right)^{n-1} = \sum_{n=1}^7 4^{2-n} = \frac{5461}{1024}$ b) $\sum_{n=1}^{10} 15(3)^{n-1} = 442860$ c) $\sum_{n=1}^{14} \left(\frac{1}{2}\right)^{n-1} = 2$

Geometric Series: Lesson 2

Part III – Applications

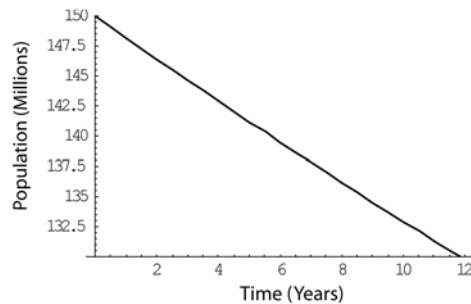
APPLICATIONS:

Example 1: The population of a country is 150 million, and is decreasing by 1.2% each year.

a) Determine an equation using an exponential function of the form $A = A_0 b^{\frac{t}{P}}$

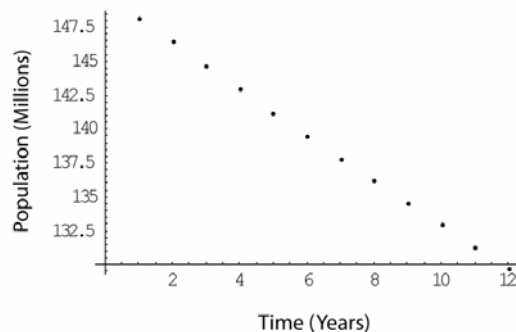
$$A = 150(0.988)^t$$

*Remember that a decreasing percent must be subtracted from 1.



b) Determine an equation using the term formula $t_n = ar^n$

$$t_n = 150(0.988)^n$$



Why $t_n = ar^n$?

If n represents term position, use $t_n = ar^{n-1}$

If n represents time, then drop the -1 and use the formula $t_n = ar^n$

c) Explain how each model must be treated differently, and explain which one is more appropriate for this situation:

In the first (exponential) model, the graph is continuous, so the amount in the middle of a year can be calculated. The second (discrete geometric) is only applicable at each full year interval. The exponential model is more appropriate since the population is continuously changing.

d) Determine the population after 10 years, 3 months using each model:

Exponential: Use $t = 10.25$ to represent 10 and a quarter years. (Answer = 132.5 million)

Geometric: Use $t = 10$ since you can't use decimals. (Answer = 132.94 million)

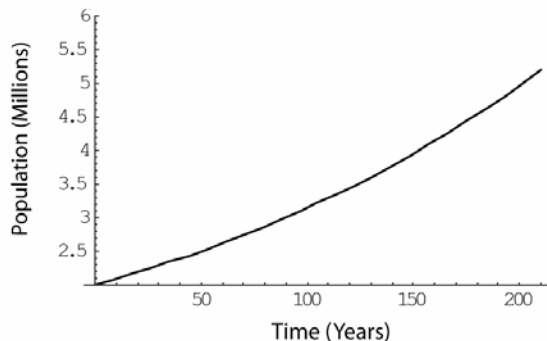
Geometric Series: Lesson 2

Part III – Applications

Example 2: In a certain country, elections are held every five years. Voter turnout in elections increases by 2.3% each time an election is held. When the country was formed in the year 1800, 2 million people voted.

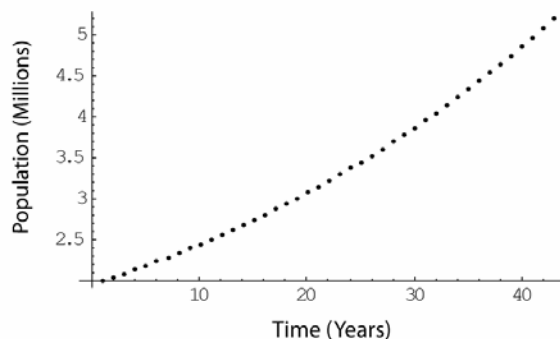
a) Determine an equation using an exponential function of the form $A = A_0 b^{\frac{t}{p}}$

$$A = 2(1.023)^{\frac{t}{5}}$$



b) Determine an equation using the term formula $t_n = ar^n$

$$t_n = 2(1.023)^n$$



c) Explain how each model must be treated differently, and which is more appropriate for this situation:

In the continuous exponential model, t is the elapsed time in years.

In the discrete geometric model, n represents the number of 5 year intervals.

The geometric model is more appropriate, since elections are a one-time event every five years.

d) Determine the number of voters in the year 2010.

Using the geometric sequence formula, there are 42 five year intervals between 1800 and 2010.

$$t_n = 2(1.023)^{42}$$

Answer = 5.20 million

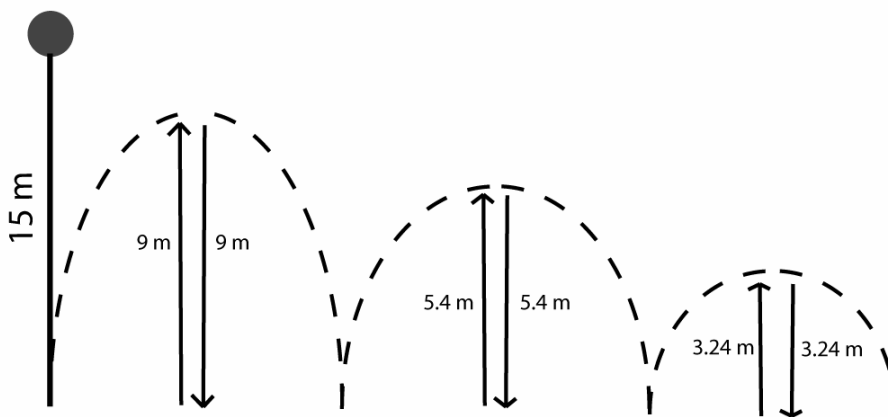
Geometric Series: Lesson 2

Part III – Applications

Example 3: A chessboard has 64 squares. If one penny is placed on the first square, then doubled to two pennies on the second, then doubled to four pennies on the third, how much money will be on the board when the 64th square is reached?

$$\begin{aligned}
 a &= 0.01 & S_n &= \frac{a(r^n - 1)}{r - 1} \\
 r &= 2 \\
 n &= 64 & S_{64} &= \frac{0.01((2)^{64} - 1)}{2 - 1} \\
 & & S_{64} &= \$1.84 \times 10^{17} = \$184,000,000,000,000,000
 \end{aligned}$$

Example 4: A ball is dropped from a height of 15 m and bounces to 60% of the previous height. How far has the ball traveled when it hits the ground for the fourth time?



$a = 15$
 $r = 0.6$
 $n = 4$

You can start this question by plugging the numbers directly into the sum formula.

$$\begin{aligned}
 S_n &= \frac{a(r^n - 1)}{r - 1} \\
 S_4 &= \frac{15((0.6)^4 - 1)}{0.6 - 1} \\
 S_4 &= 32.64 \text{ m}
 \end{aligned}$$

The sum just calculated, by plugging in numbers directly from the question, is **NOT THE ANSWER!**

You need to do two extra things to get the answer:

- 1) Since the ball goes up and down, multiply the sum by 2 to get the total distance.
- 2) Now subtract 15, since the first height only has a down motion.

The total distance is **50.28 m**

Geometric Series: Lesson 2

Part III – Applications

QUESTIONS:

1) A person puts \$1000 in a safe. Each year, they put in 15% of what's already in the safe.

- Determine an equation using an exponential function of the form $A = A_0 b^{\frac{t}{P}}$
- Determine an equation using the sum formula of the form $t_n = ar^n$
- Determine the value in the safe in 10 years, 6 months using each model.
- Which model is more appropriate for this situation?

2) An oil well produces 13500 m³ of oil, but the production is dropping by 3% each year.

- Determine an equation using an exponential function of the form $A = A_0 b^{\frac{t}{P}}$
- Determine an equation using the sum formula of the form $t_n = ar^n$
- Determine the amount of oil remaining after 3 years.
- Which model is more appropriate for this situation?

3) A gameboard has 32 squares. If one nickel is placed on the first square, then doubled to two nickels on the second, then doubled to four nickels on the third, how much money will be on the board when the 32nd square is reached?

4) A ball is dropped from a height of 17 m and bounces to 68% of the previous height. How far has the ball traveled vertically when it hits the ground for the eighth time?

5) A ball is dropped from a height of 20 m and bounces to 4/5 of the previous height. How far has the ball traveled vertically when it hits the ground for the tenth time?

ANSWERS:

1) a) $A = 1000(1.15)^t$

b) $t_n = 1000(1.15)^n$

c) $A = 1000(1.15)^t = 1000(1.15)^{10.5} = 4338$

$t_n = 1000(1.15)^{10} = 4045.56$

d) Use the geometric model since money is placed in the safe only once per year.

2) a) $A = 13500(0.97)^t$

b) $t_n = 13500(0.97)^n$

c) $A = 13500(0.97)^t = 13500(0.97)^3 = 12321.09$

$t_n = 13500(0.97)^{n-1} = 13500(0.97)^3 = 12321.09$

d) Even though the answers are the same, use the exponential model since the oil well is operated continuously.

3) \$214748364.40

4) 84.4

5) 158.53